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EXTENDING OPERATORS INTO LINDENSTRAUSS SPACES

BY

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ABSTRACT

We study the global and local approaches to the problem of extension of operators into Lindenstrauss spaces.

1. Introduction

Early in the 70's Lindenstrauss and Pełczyński proved in [20] that every $C(K)$-valued operator from a subspace of c_0 can be extended to the whole c_0. In that paper it is remarked that the same holds true replacing "$C(K)$-space" by "isometric L_1-predual", later called Lindenstrauss spaces in the literature. The result therefore should be:

Theorem 1.1: *Let X be a closed subspace of c_0. Let be Y a Banach space such that $Y^* = L_1(\mu)$ for some measure μ and let $T : X \to Y$ be an operator. Then, for each $\varepsilon > 0$, T admits an extension to an operator $\widetilde{T} : c_0 \to Y$ with $\|\widetilde{T}\| < (1+\varepsilon)\|T\|$.*

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A proof for this result, which to the best of our knowledge has never appeared explicitly in the literature, is presented in the Appendix.

The paper thus focuses on the study of the extension of operators into Lindenstrauss spaces. Following Zippin's language, in Section 2 we adopt a global approach to describe how the extension into different types of Lindenstrauss spaces comes characterized by the existence of different types of weak*-continuous selectors. Sections 3 and 4 adopt a local approach; with it, we prove the Lindenstrauss–Pełczyński result, extend it to the nonseparable case, provide new constructions in which the extension of operators into Lindenstrauss spaces exists and show that the Bourgain–Pisier construction [5] of exotic $\mathcal{L}_\infty$-spaces is one of those. Section 5 shows the difference between extending operators into $C(K)$-spaces, into Lindenstrauss spaces, into $\mathcal{L}_\infty$-spaces that already admit extension of operators from subspaces of c_0 and into arbitrary $\mathcal{L}_\infty$-spaces. Section 6 is the Appendix with the proof of the Lindenstrauss–Pełczyński result following the indication of the authors of [20].

2. The global approach to the extension problem into Lindenstrauss spaces

The following definition taken from [11] shall be useful:

Definition: Given a class $\mathcal{A}$ of Banach spaces and a positive scalar λ we will say that an exact sequence $0 \to Y \to X \to Z \to 0$ is $(\lambda, \mathcal{A})$-trivial or that $(\lambda, \mathcal{A})$-splits if and only if for every $A \in \mathcal{A}$ every operator $\tau : Y \to A$ can be extended to an operator $T : X \to A$ verifying $\|T\| \leq \lambda\|\tau\|$. When it is not necessary to specify the λ we shall simply say that the sequence $\mathcal{A}$-splits or that it is $\mathcal{A}$-trivial.

A Banach space is said to be a Lindenstrauss space if its dual is isometric to some $L_1(\mu)$-space. In what follows we will denote by $\mathfrak{L}$ the class of Lindenstrauss spaces. The following subclasses of $\mathfrak{L}$ were isolated in [22, 19]:

- $\mathcal{C}(\mathcal{K})$**:** The spaces of continuous functions on compact Hausdorff spaces.
- $\mathcal{C}(\mathcal{K})_0$**:** The spaces of continuous functions on compact Hausdorff spaces K which vanish at a fixed point.
- $\mathcal{C}_\sigma(\mathcal{K})$**:** The spaces of continuous functions on compact Hausdorff spaces K which satisfy $f(\sigma k) = -f(k)$ for all $k \in K$, where $\sigma : K \to K$ is a homeomorphism of period 2.

$\mathcal{C}(\mathcal{K})_\Sigma$**:** Those $C_\sigma(K)$ spaces for which σ has no fixed points.

$\mathcal{M}$**:** Sublattices of $C(K)$ spaces. What is the same, spaces X which can be represented as follows: There exist a Hausdorff compact space K and a set of triples $\{k^1_\alpha, k^2_\alpha, \lambda_\alpha\}_{\alpha\in A}$ with $k^1_\alpha, k^2_\alpha \in K$ and $\lambda_\alpha \geq 0$ such that X is the set of all $f \in C(K)$ which satisfy $f(k^1_\alpha) = \lambda_\alpha f(k^2_\alpha)$ for all $\alpha \in A$.

$\mathcal{G}$**:** Spaces defined like the explicit definitions of $\mathcal{M}$-spaces only that now the λ_α are allowed to be arbitrary real numbers.

$\mathcal{A}(\mathcal{S})$**:** Spaces of affine functions on a simplex S.

The global approach to the extension problem was introduced by Zippin [29] as follows:

Lemma 2.1: *Given a subspace $j : Y \to X$ of a Banach space, every operator $T : Y \to C(K)$ can be extended to an operator $\widehat{T} : X \to C(K)$ with estimate $\|\widehat{T}\| \leq \lambda\|T\|$ if and only if there is a weak*-continuous map $\omega : B_{Y^*} \to \lambda B_{X^*}$ such that $j^*\omega = id$.*

The map ω shall be called a weak*-continuous selector or simply a w^*-selector for j^* (a λw^*-selector if the quantitative estimate is needed). Given a weak*-continuous selector ω for an isometric embedding $j : Y \to X$, every operator $\tau : Y \to C(K)$ can be extended through j by the formula

$$\tau^\omega(x)(k) = \langle \omega\tau^*(k), x\rangle.$$

The operator τ^ω defined in this way will be called the Zippin extension of τ through j using ω. It is clear that

$$\|\tau^\omega\| \leq \|\omega\|\|\tau\|$$

where $\|\omega\| = \sup\{\|\omega(y^*)\| : \|y^*\| \leq 1\}$. Conversely, if every operator $Y \to C(K)$ can be extended to X through j then one just needs to extend the canonical embedding $\delta : Y \to C(B_{Y^*})$ to an operator $\Delta : X \to C(B_{Y^*})$, to obtain a weak*-continuous selector ω for j^*:

$$\langle \omega(y^*), x\rangle = \Delta(x)(y^*)$$

that verifies $\|\omega\| \leq \|\Delta\|$.

Zippin uses this criterion in [29, 30] to obtain different proofs of the Lindenstrauss–Pełczyński theorem, and in [28] to embed every separable Banach space X into some separable space Z_X with FDD in such a way that Z_X/X also has FDD and moreover the sequence $0 \to X \to Z_X \to Z_X/X \to 0$ is $\mathcal{C}(\mathcal{K})$-trivial.

There are simple correspondences between the type of Lindenstrauss space that can be set as target space and the type of w^*-selector that is possible to obtain. We collect all the information in the following omnibus lemma.

Lemma 2.2: *Let $0 \to Y \xrightarrow{j} X \to Z \to 0$ be an exact sequence.*

1. *It is $(\lambda, \mathcal{C}(\mathcal{K}))$-trivial if and only if there is a λw*-selector for j^*.*
2. *It is $(\lambda, \mathcal{C}(\mathcal{K})_0)$-trivial if and only if there is a λw*-selector for j^* such that $\omega(0) = 0$. In particular, it is $(\lambda, \mathcal{C}(\mathcal{K})_0)$-trivial if and only if it is $(\lambda, C(K))$-trivial.*
3. *It is $(\lambda, \mathcal{C}_\sigma(\mathcal{K}))$-trivial if and only if there is a symmetric λw*-selector for j^*. In particular, it is $(\lambda, \mathcal{C}_\sigma(\mathcal{K}))$-trivial if and only if it is $(\lambda, C(K))$-trivial.*
4. *$(\lambda, \mathcal{M})$-trivial if and only if there is a positive homogeneous λw*-selector for j^*.*
5. *$(\lambda, \mathcal{G})$-trivial if and only if there is a homogeneous λw*-selector for j^*. If Y is separable, $\mathcal{G}$-splitting coincides with $C(K)$-splitting.*
6. *$(\lambda, \mathcal{A}(\mathcal{S}))$-trivial for a fixed $\tau : Y \to \mathcal{A}(\mathcal{S})$ if and only if there exists a weak* continuous map $\omega : S \to \lambda B_{X^*}$, such that $j^*\omega = \tau^*\delta$.*
7. *Trivial if and only if there is a affine w^*-selector for j^*.*

Proof. The proof of (2) and (3) is essentially the same. In both cases, the first part is routine while for the second, given a selector ω, the symmetric map $\widetilde{\omega}(x) := \frac{\omega(x)-\omega(-x)}{2}$ is also a selector. Moreover, the equivalence between $C(K)_0$-splitting and $C(K)$-splitting is clear since $C(K)_0$-spaces are at worst 2-complemented subspaces of $C(K)$-spaces. It was proved by Samuel [25] that separable $\mathcal{C}_\sigma(\mathcal{K}))$-spaces are isomorphic to $C(K)$-spaces, which yields the equivalence between $\mathcal{C}_\sigma(\mathcal{K}))$-splitting and $C(K)$-splitting for separable Y.

We prove now the case of $\mathcal{G}$-spaces which is by far the most interesting. Observe first that the space $G(B_{Y^*}) = \{f \in C(B_{Y^*}) : f(\lambda y^*) = \lambda f(y^*)\}$ of homogeneous weak*-continuous functions on B_{Y^*} endowed with the supremum norm is a $\mathcal{G}$-space constructed over the set of triples

$$\{\{\lambda y^*, y^*, \lambda\}_{\lambda \in \lambda(y^*)}\}_{y^* \in B_{Y^*}}$$

where $\lambda(y^*)$ is $\{\lambda \in \mathbb{R} : \lambda y^* \in B_{Y^*}\}$ for every $y^* \in B_{Y^*}$. Moreover, the canonical embedding $\delta : Y \to G(B_{Y^*})$ has the universal property that every operator $\tau : Y \to G$ into any $\mathcal{G}$-space can be extended to $G(B_{Y^*})$ through δ. Indeed, assume that $A = \{a_\alpha, b_\alpha, \lambda_\alpha\}_\alpha$ is the set of triples for G with base compact space K, and let $\tau : Y \to G$ be a norm-one operator. We define $T : G(B_{Y^*}) \to G$ as usual $T(f)(k) = f(\tau^* k)$. We only have to check that T takes values in G. Since τ takes values in G, $\tau^*(a_\alpha) = \lambda_\alpha \tau^*(b_\alpha)$, and therefore

$$Tf(a_\alpha) = f(\tau^* a_\alpha) = f(\lambda_\alpha \tau^*(b_\alpha)) = \lambda_\alpha Tf(b_\alpha).$$

It is clear that $T\delta = \tau$

Therefore, given an embedding $j : Y \to X$, every operator $\tau : Y \to G$ into a $\mathcal{G}$-space extends through j if and only if δ extends through j. Now, assume that δ admits an extension $\Delta : X \to G(B_{Y^*})$ through j. Then

$$\omega(y^*)(x) = \Delta(x)(y^*)$$

is an homogeneous $\|\Delta\| w^*$-selector for j^*. It is homogeneous since

$$\omega(\lambda y^*)(x) = \Delta(x)(\lambda y^*) = \lambda \Delta(x)(y^*) = \lambda \omega(y^*)(x).$$

Conversely, if there is an homogeneous λw^*-selector ω for j^* then $\delta(x)(y^*) = \omega(y^*)(x)$ is an extension of δ with norm at most λ. ∎

Now we present a result showing how often homogeneous selectors appear. The interest in the existence of this particular class of selectors has been recently renowned by works of Kalton, see [17] and [18]. It is also the key to prove the nonseparable version of Lindenstrauss–Pełczyński's extension theorem as is done by Johnson and Zippin [15].

Theorem 2.1: *Let Y be a separable subspace of X. If there exists a weak*-continuous selector $B_{Y^*} \to \lambda B_{X^*}$ then there is a homogeneous weak*-continuous selector $B_{Y^*} \to 3\lambda B_{X^*}$.*

Proof. The key is Benyamini's proof [1] that every separable $\mathcal{G}$-space is isomorphic to a $C(K)$-space. Thus, let $\alpha : G(B_{Y^*}) \to C(K)$ be an isomorphism. If there is a λw^*-selector ω for j^* then $\alpha\delta$ can be extended to an operator $\overline{\alpha\delta} : X \to C(K)$. Therefore $\alpha^{-1}\overline{\alpha\delta} : X \to G(B_{Y^*})$ is an extension of δ through j. The homogeneous weak*-continuous selector is then $\omega'(y^*)(x) = \alpha^{-1}\overline{\alpha\delta}(x)(y^*)$. The value of the new constant is, at first sight, $\mu = \lambda \inf d(C(K), G(B_{Y^*}))$ where d denotes the Banach–Mazur distance, and the inf is taken over all $C(K)$ spaces. To get the precise estimate of the constant we need to go inside Benyamini's proof. Benyamini proves there exists an isomorphism $S : G(B_{Y^*}) \to Z$ where Z is isometric to $C_\sigma(S)$ for some compact metric S. This isomorphism is controlled by $\|S\| \leq 3/2$ and $\|S^{-1}\| \leq 2$. To conclude just apply (3) in the lemma above. ∎

In the nonseparable case, Benyamini [2] constructed an $\mathcal{M}$-space that is not complemented in any $C(K)$-space. If we call M to such space the sequence $0 \to M \to C(B_{M^*}) \to Z \to 0$ is obviously $\mathcal{C}(\mathcal{K})$-trivial but not $\mathcal{M}$-trivial and therefore not $\mathcal{G}$-trivial. This means that in this case there exists a weak*-continuous selector but not an homogeneous weak*-continuous selector. In other words, $C(K)$-splitting does not imply in general $\mathcal{M}$-splitting although in the separable setting $\mathcal{C}(\mathcal{K})$-splitting and $\mathcal{G}$-splitting coincide. Zippin showed in [28, Ex.3] that for $1 < p < +\infty$ every exact sequence $0 \to W \to l_p \to l_p/W \to 0$ is $(1, \mathcal{C}(\mathcal{K}))$-trivial. It has been proved now that they are $\mathcal{G}$-trivial (actually $(1, \mathcal{G})$-trivial if one observes that the weak*-continuous selector that Zippin provides is itself homogeneous).

3. The local approach to the extension problem into Lindenstrauss spaces

Let us briefly sketch the push-out construction in Banach spaces since it is essential for our purposes; it can be seen described in full details, in the category of Banach spaces, in [9, 6].

Given an operator $S : Y \to M$ and an embedding $j : Y \to X$, their push-out is the quotient space $PO = M \oplus_1 X/\overline{\Delta}$ where $\Delta = \{(Sy, -jy) \in M \oplus_1 X\}$. There exist two operators: $u_S : X \to PO$ and $u_j : M \to PO$ such that $u_S j = u_j S$, which are the restrictions to M and X of the quotient map $M \oplus_1 X \to PO$. The push-out construction has the universal property that given two operators $\alpha : M \to E$ and $\beta : X \to E$ such that $\alpha S = \beta j$ there exists a unique operator $\gamma : PO \to E$ such that $\gamma u_j = \alpha$ and $\gamma u_S = \beta$ and moreover $\|\gamma\| \leq \max\{\|\alpha\|, \|\beta\|\}$.

Let us also recall that an exact sequence $0 \to Y \xrightarrow{j} X \to Z \to 0$ of Banach spaces is a diagram in which the kernel of each arrow coincides with the image of the preceding. An exact sequence is said to split if there is a linear continuous projection of X onto $j(Y)$; namely, the identity of Y can be extended through j to a linear continuous operator $X \to Y$. We shall use the notation $0 \to Y \xrightarrow{j} X \to Z \to 0 \equiv F$ when a name for the sequence becomes necessary. If $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$ is an exact sequence then with $Q[(m, x) + \Delta] = qx$ there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{j} & X & \xrightarrow{q} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle u_S} & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow[u_j]{} & PO & \xrightarrow[Q]{} & Z & \longrightarrow & 0. \end{array}$$

THEOREM 3.1: *Let H be a closed subspace of c_0. Let be Y a Banach space in which every separable subspace is contained in the inductive limit of a sequence*

$$Y_1 \xrightarrow{\delta_1} Y_2 \xrightarrow{\delta_2} Y_3 \xrightarrow{\delta_3} \cdots$$

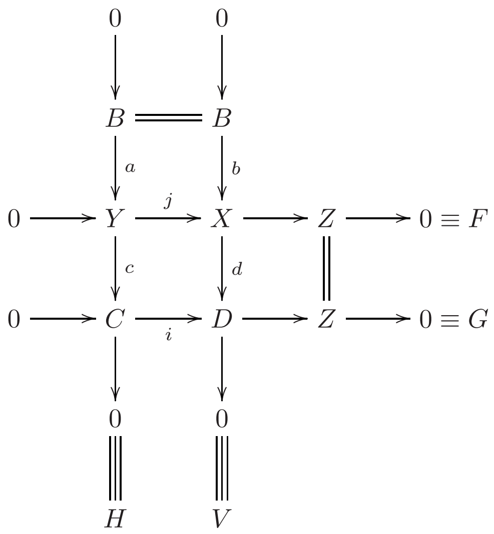
of subspaces of Y in which Y_n is $(1+2^{-n})$-isomorphic to some finite-dimensional $l_\infty^{k(n)}$-space, and the operators δ_n are into isometries. Then, for each $\varepsilon > 0$ every operator $T : X \to Y$ admits an extension to an operator $\widetilde{T} : c_0 \to Y$.

Proof. There is no loss of generality assuming that Y is separable since the range of T actually lies on a separable subspace of Y; and, therefore, that Y itself is an inductive limit as in the statement of the theorem. Our second set of simplifications is:

- To assume that $H = c_0(A_n)$, where (A_n) is a sequence of finite-dimensional Banach spaces.
- To assume that $c_0(A_n)$ is embedded into c_0 in the following specific form: let $a_n : A_n \to l_\infty^{a(n)}$ be an $(1 + 2^{-n})$-isometry and then consider the embedding $j_0 : c_0(A_n) \to c_0(l_\infty^{a(n)})$ given by $j_0((x_n)) = (a_n(x_n))$.

It follows from Sobczyk's theorem that if every operator $T : c_0(A_n) \to Y$ can be extended to c_0 through j_0 then every operator $T : c_0(A_n) \to Y$ can be extended to c_0 through any embedding. That it is enough to work with subspaces having the form $c_0(A_n)$ for a certain collection (A_n) of finite dimensional Banach spaces follows from classical results of Johnson-Rosenthal and Zippin (see [21]) asserting that each subspace H of c_0 admits a subspace having the form $c_0(A_n)$ such that the corresponding quotient space $H/c_0(A_n)$ has the form $c_0(B_n)$. The rest is a 3-space-like argument:

Lemma 3.1: *Let $\mathcal{A}$ be a class of Banach spaces. Consider the completed push-out diagram of Banach spaces*



Then V and G are $\mathcal{A}$-trivial if and only if F and H are $\mathcal{A}$-trivial.

Proof. Assume that V and G are $\mathcal{A}$-trivial. It is clear that if V is $\mathcal{A}$-trivial then so is H. Let us show that also F is $\mathcal{A}$-trivial. Let $A \in \mathcal{A}$ and $\tau : Y \to A$ an operator. Since V is $\mathcal{A}$-trivial, τa can be extended to an operator T_X through b. Since $(\tau - T_X j)a = 0$, there is $T_C : C \to A$ such that $T_C c = \tau - T_X j$. Since G is $\mathcal{A}$-trivial, T_C can be extended to an operator T_D through i. The operator $T_X + T_D d : X \to A$ is the desired extension:

$$(T_X + T_D d)j = T_X j + T_D ic = T_X j + T_C c = T_X j + \tau - T_X j = \tau.$$

Assume now that H and F are $\mathcal{A}$-trivial. Then G is necessarily $\mathcal{A}$-trivial by the universal property of the push-out. It remains to show that V is $\mathcal{A}$-trivial. Let $\tau : B \to A$. Take T_Y an extension of τ through a and then T_X an extension of T_Y through j. This is the desired extension since $T_X b = T_X ja = T_Y a = \tau$. ∎

Returning to the main proof, let then $\phi : c_0(A_n) \to Y$ be a norm one operator. The construction of the extension $\Phi : c_0(l_\infty^{a(n)}) \to Y$ of ϕ shall be performed inductively. We actually will define Φ on the dense subspace of finitely supported elements. In this way, everything reduces to a question involving finite dimensional Banach spaces. Let $\varepsilon_n > 0$ be such that $\prod(1+\varepsilon_n) \leq 1+\varepsilon$.

Let $a_n : A_n \to l_\infty^{a(n)}$ be a $(1+\varepsilon_n)$-isometric embedding and let $\delta_n : l_\infty^{k(n)} \to l_\infty^{k(n+1)}$ be an isometric embedding. There is no loss of generality assuming that the restriction ϕ_n of ϕ to $A_1 \oplus_\infty \cdots \oplus_\infty A_n$ lies in $Y_n = l_\infty^{k(n)}$. Let $\omega_n : B_{A_n^*} \to (1+\varepsilon_n)B_{l_1^{a(n)}}$ be a homogeneous weak*-continuous selector for a_n^* (it exists thanks to the Bartle-Graves continuous selection (see [3] or else [9]). Let Φ_1 be the operator $(\delta_1\phi_1)^{\omega_1} : l_\infty^{a(1)} \to l_\infty^{k(2)}$, a Zippin extension of $\delta_1\phi_1$ through a_1 using ω_1. We set the operator

$$\Phi_2 = (\delta_1\phi_1)^{\omega_1} \oplus \phi_2 : l_\infty^{a(1)} \oplus_\infty A_2 \longrightarrow l_\infty^{k(2)}$$

which is defined as

$$\left[(\delta_1\phi_1)^{\omega_1} \oplus \phi_2\right](x,y) = (\delta_1\phi_1)^{\omega_1}(x) + \phi_2(y).$$

Observe that $[(\delta_1\phi_1)^{\omega_1} \oplus \phi_2](a_1, 1_{A_2}) = \phi_2$ since $(\delta_1\phi_1)^{\omega_1}a_1 = \delta_1\phi_1 = \phi_2(1_{A_1}, 0)$. Moreover,

$$(\delta_1\phi_1)^{\omega_1} \oplus \phi_2 = \phi_2^{(\omega_1,1)},$$

where $(\omega_1, 1) : B_{A_1^* \oplus_1 A_2^*} \to (1+\varepsilon_1)B_{l_1^{a(n)} \oplus_1 A_2^*}$ is the homogeneous weak*-continuous selector $(\omega_1,1)(a_1^*, a_2^*) = (\omega_1(a_1^*), a_2^*)$. It therefore follows

$$\|(\delta_1\phi_1)^{\omega_1} \oplus \phi_2\| \leq (1+\varepsilon_1)\|\phi_2\|.$$

Assuming the operator $\Phi_n : l_\infty^{a(1)} \oplus_\infty \cdots \oplus_\infty l_\infty^{a(n-1)} \oplus_\infty A_n \longrightarrow l_\infty^{k(n)}$ has already been constructed verifying $\|\Phi_n\| \leq (1+\varepsilon_1)\cdots(1+\varepsilon_n)\|\phi_n\|$ and $\Phi_n(a_1, \ldots, a_{n-1}, 1_{A_n}) = \phi_n$ then the operator

$$\Phi_{n+1} = (\delta_n\Phi_n)^{\omega_n} \oplus \phi_{n+1} : l_\infty^{a(1)} \oplus_\infty \cdots \oplus_\infty l_\infty^{a(n)} \oplus_\infty A_{n+1} \longrightarrow l_\infty^{k(n+1)}$$

comes defined as

$$\left[(\delta_n\Phi_n)^{\omega_n} \oplus \phi_{n+1}\right](x,y) = (\delta_n\Phi_n)^{\omega_n}(x) + \phi_{n+1}(y).$$

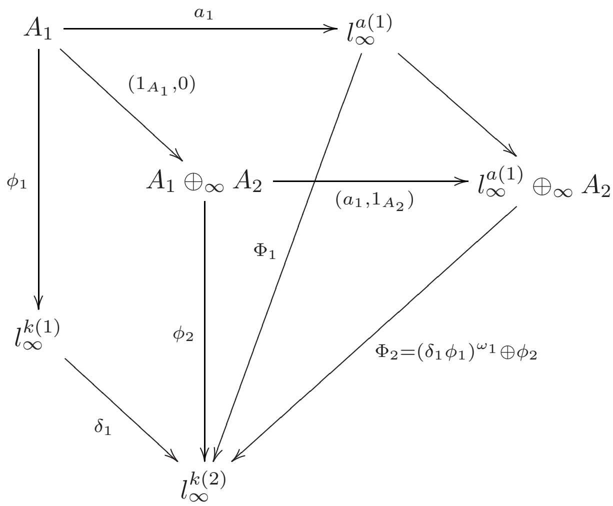
Reasoning exactly as in step 1, one has

$$\|\Phi_{n+1}\| \leq (1+\varepsilon_1)\cdots(1+\varepsilon_n)(1+\varepsilon_{n+1})\|\phi_{n+1}\|$$

and

$$\Phi_{n+1}(a_1 \oplus \cdots \oplus a_n \oplus 1_{A_{n+1}}) = \phi_{n+1}.$$

The process is illustrated by the following diagram.



To keep track of the norm of the extension is not easy: the inductive process yields an estimate of $\prod(1+\varepsilon_n) \leq (1+\varepsilon)$; which Sobczyk's theorem doubles when one considers an arbitrary embedding. It is the 3-space argument which spoils the estimate.

After the work of Michael and Pełczyński [23] and then Lazar and Lindenstrauss [19], separable isometric preduals of L_1 are precisely inductive limits of finite dimensional spaces F_n such that F_n can be chosen $1+2^{-n}$-isomorphic to $l_\infty^{dim F_n}$. ∎

The local approach allows us to obtain the nonseparable version for the extension into Lindenstrauss spaces.

Theorem 3.2: *Every sequence* $0 \to H \to c_0(\Gamma) \to c_0(\Gamma)/H \to 0$ *is* $\mathfrak{L}$*-trivial.*

Proof. We first need a result of Moreno and Plichko [24] providing a decomposition of Γ as $\cup_{\alpha\in A}\Gamma_\alpha$ in countable sets Γ_α and three isometries $u : H \to c_0(H_\alpha)$, $v : c_0(\Gamma) \to c_0(c_0(\Gamma_\alpha))$ and $w : c_0(\Gamma)/H \to c_0(c_0(\Gamma_\alpha)/H_\alpha)$ in such a way that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & c_0(\Gamma) & \longrightarrow & c_0(\Gamma)/H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & c_0(H_\alpha) & \longrightarrow & c_0(c_0(\Gamma_\alpha)) & \longrightarrow & c_0(c_0(\Gamma_\alpha)/H_\alpha) & \longrightarrow & 0 \end{array}$$

is commutative. Here the exact sequence below is the c_0-amalgam of certain exact sequences $0 \to H_\alpha \to c_0(\Gamma_\alpha) \to c_0(\Gamma_\alpha)/H_\alpha \to 0$. It is therefore sufficient to prove the result for the lower sequence.

Given thus an operator $\phi : c_0(H_\alpha) \to Y$, let us consider the restrictions $\phi_\alpha : H_\alpha \to Y$ and then extend each to $\Phi_\alpha : c_0(\Gamma_\alpha) \to Y$ with a uniform bound on the norm of the extensions. It remains to show that the amalgam of the extensions $\Phi(x_\alpha) = \sum \Phi_\alpha(x_\alpha)$ defines an operator $c_0(c_0(\Gamma_\alpha)) \to Y$. This happens, see [4], if and only if there is a norming subset $N \subset B_{Y^*}$ such that

$$\sup_{y^* \in N} \sum \|\Phi_\alpha^*(y^*)\| < +\infty. \tag{1}$$

In our case, since each H_α is separable, the inductive approach we have just described applies to extend ϕ_α. We omit from now on the subindex α. To prove that condition (1) holds, it is sufficient to show that $\sup \sum \|\Phi^*(y^*)\| < +\infty$ when y^* is any of the extreme points $e^*_{j,k(n)}$ of $(l_\infty^{k(n)})^*$ to which Φ^* eventually applies. What we actually show is that

$$\|\Phi^*(e^*_{j,k(n)})\| \leq C\|\phi^*(e^*_{j,k(n)})\|. \tag{2}$$

To prove this, let us observe that a Zippin extension of an operator $\tau : Y \to C(K)$ through an embedding $j : Y \to X$ using a homogeneous weak*-continuous selector ω verifies:

$$\|(\tau^\omega)^*(k)\| \leq \|\omega\|\|\tau^*(k)\|.$$

Since $(\delta_n\Phi_n)^{\omega_n} \oplus \phi_{n+1} = \phi_{n+1}^{(\omega_n,1)}$ is a Zippin extension of ϕ_{n+1} using the homogeneous weak*-continuous selector $(\omega_n, 1)$ it verifies

$$\|\Phi_n^*(e^*_{j,k(n)})\| \leq (1+\varepsilon_n)\|\phi_n^*(e^*_{j,k(n)})\|,$$

from where the estimate follows. The passing through Sobczyk's theorem is entirely harmless, except for doubling the constant. ∎

A warning here is in order: if one could combine the Moreno-Plichko decomposition with a proof about the extension of $\mathcal{L}$-valued operators yielding an estimate $1+\varepsilon$ (such as the one appearing in the appendix) then the final estimate here would be $1+\varepsilon$. The $6+\varepsilon$ is a legacy of our method of proof for Theorem 3.1.

The nonseparable version for the extension into $C(K)$-spaces was obtained using global arguments by Johnson and Zippin in [15]; it requires to be combined with Theorem 1.1. in [30]. See [31] for a detailed exposition. Observe that a global proof for the extension of operators into Lindenstrauss spaces is not possible until a global characterization is available.

Problem: *Does there exist a global characterization for the extension of operators into Lindenstrauss spaces?*

4. Universal $\mathfrak{L}$–trivial sequences, and new examples

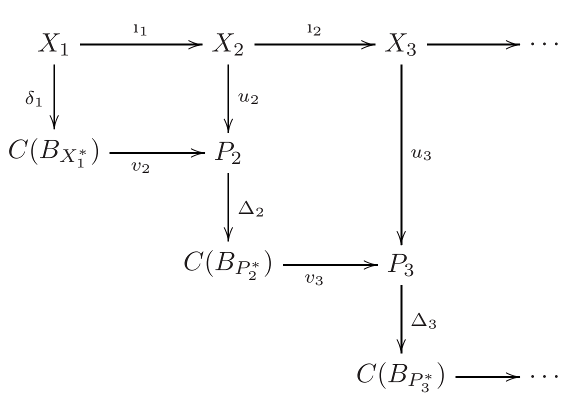
A first step toward solving the previous problem is to obtain an embedding $X \to \mathfrak{L}^1(X)$ which is, regarding the extension of operators into Lindenstrauss spaces, as "universal" as it is the canonical embedding $X \to C(B_{X^*})$ with respect to the extension into $C(K)$-spaces. The construction of the space $\mathcal{L}^1(X)$ and the universal embedding are the contents of the next result.

Proposition 4.1 (Universal construction, isometric version): *Given a Banach space X there exists a Lindenstrauss superspace $\mathcal{L}^1(X)$ such that the exact sequence*

$$0 \to X \to \mathcal{L}^1(X) \to \mathcal{L}^1(X)/X \to 0$$

is $(1, \mathfrak{L})$-trivial. If X is separable then $\mathfrak{L}^1(X)$ can be chosen separable as well.

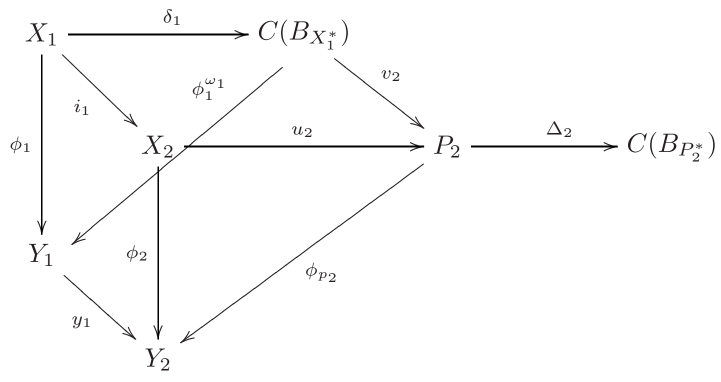
Proof. **The separable case.** Assume that X is separable and let us represent it as $\overline{\cup X_n}$ in which each X_n is finite-dimensional, with embeddings $i_n : X_n \to X_{n+1}$. Next, let us consider the canonical isometric embedding $\delta_1 : X_1 \to C(B_{X_1^*})$, the isometric embedding $i_1 : X_1 \to X_2$ and obtain the push-out space P_2. Let $\Delta_2 : P_2 \to C(B_{P_2^*})$ denote the canonical embedding. Observe the diagram



in which P_3 is the push-out space of the embeddings $\Delta_2 u_2 : X_2 \to P_2$ and $i_2 : X_2 \to X_3$. The process continues inductively. The Lindenstrauss space we are looking for is the inductive limit

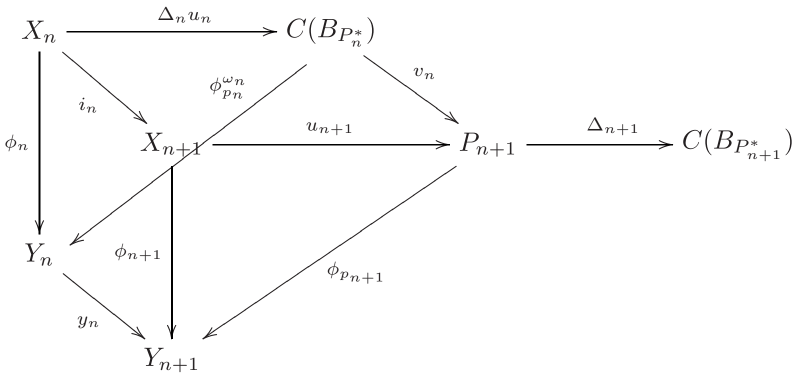
$$C(B_{X_1^*}) \xrightarrow{\Delta_2 v_2} C(B_{P_2^*}) \xrightarrow{\Delta_3 v_3} C(B_{P_3^*}) \longrightarrow \cdots \equiv \mathcal{L}^1(X).$$

while the embedding $J : X \longrightarrow \mathcal{L}^1(X)$ is locally defined by the vertical isometric embeddings $\Delta_n u_n : X_n \to C(B_{P_n^*})$. We show that this construction has the universal property that every operator $T : X \to \mathcal{L}$ into a Lindenstrauss space can be extended through J. Let $\mathcal{L}^1$ be a separable Lindenstrauss space which, using [19] can be put as an inductive limit $\lim_{\rightarrow} Y_n$ (with isometric embeddings y_n) in which Y_n is isometric to some $l_\infty^{k(n)}$ space. Let $\phi : X \to \mathcal{L}^1$ be a norm one operator for which we assume that $\phi_n = \phi_{|X_n}$ sends X_n into Y_n. For the first step, consider the diagram



Here $\phi_1^{\omega_1}$ is an extension of ϕ_1 obtained through the canonical homogeneous 1-weak*-continuous selector for δ_1^*. Since Y_1 is isometric to $l_\infty^{k(1)}$ we have $\|\phi_1^{\omega_1}\| \le \|\phi_1\|$. The universal property of the push-out yields a (unique) operator $\phi_{p_2} : P_2 \to Y_2$ verifying $\phi_{p_2} u_2 = \phi_2$ and $\phi_{p_2} v_2 = y_1 \phi_1^{\omega_1}$ and such that $\|\phi_{p_2}\| \le \max\{\|y_1 \phi_1^{\omega_1}\|, \|\phi_2\|\} \le 1$. Finally, take as $\phi_{p_2}^{\omega_2}$ an extension of ϕ_{p_2}

obtained through the canonical homogeneous 1-weak*-continuous selector for Δ_2^*. For the general case, consider the diagram



Here $\phi_{p_n}^{\omega_n}$ is an extension of ϕ_{p_n} obtained through the canonical homogeneous 1-weak*-continuous selector for Δ_n^*.

Since Y_n is isometric to $l_n^{k(n)}$ we have $\|\phi_{p_n}^{\omega_n}\| \le \|\phi_{p_n}\|$. The universal property of the push-out yields a (unique) operator $\phi_{p_{n+1}} : P_{n+1} \to Y_{n+1}$ verifying $\phi_{p_{n+1}} u_{n+1} = \phi_{n+1}$ and $\phi_{p_{n+1}} v_{n+1} = y_n \phi_{p_n}^{\omega_n}$ and such that

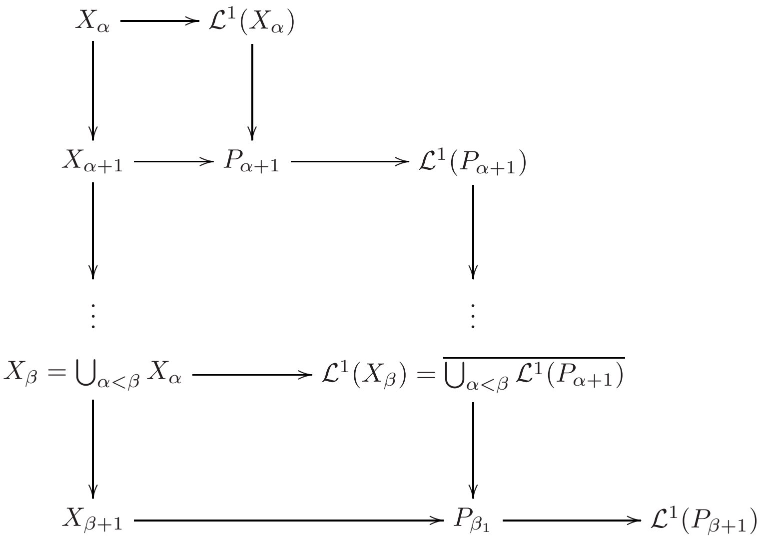
$$\|\phi_{p_{n+1}}\| \le \max\{\|y_n \phi_{p_n}^{\omega_n}\|, \|\phi_{n+1}\|\} \le 1.$$

Finally, $\phi_{p_{n+1}}^{\omega_{n+1}}$ is an extension of $\phi_{p_{n+1}}$ obtained through the canonical homogeneous 1-weak*-continuous selector for Δ_{n+1}^*. Since Y_{n+1} is isometric to $l_n^{k(n+1)}$ we have $\|\phi_{p_{n+1}}^{\omega_{n+1}}\| \le \|\phi_{p_{n+1}}\|$.

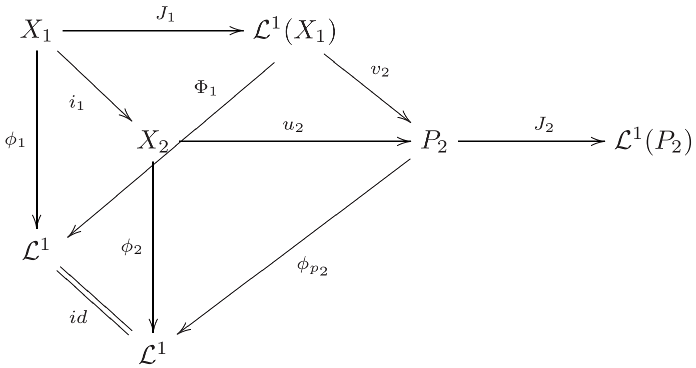
Therefore, if $\Phi : \mathcal{L}^1(X) \to \mathcal{L}^1$ denotes the operator locally defined as $\Phi_{|C(B_{P_n^*})} = \phi_{p_n}^{\omega_n}$ then one has

$$\|\Phi\| \le \sup_n \|\phi_{p_{n+1}}^{\omega_{n+1}}\| \le \|\phi_1\| \le \|\phi\|.$$

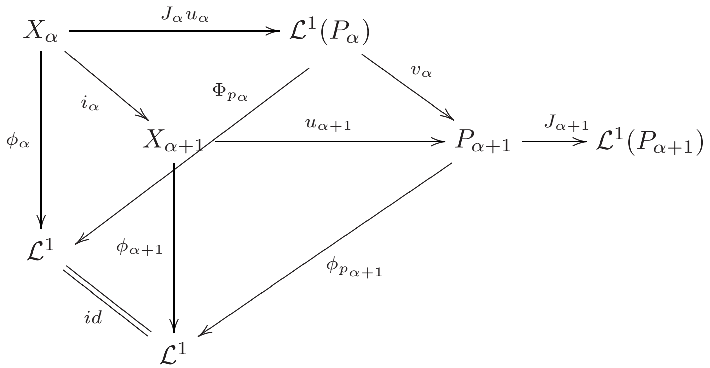
Nonseparable case. We perform a transfinite induction with some variation in the construction with respect to the separable case. It is clear that a Banach space with $\mathrm{dens}(X) = \aleph_1$ can be represented as an inductive limit $X = \lim_{\to} X_\alpha$ of separable spaces ordered by the family of all countable ordinals α. Since the range of an operator defined on a separable space into a Lindenstrauss space lies into a separable Lindenstrauss space, the argument for the separable case can be continued to get an universal embedding $J : X \to \mathcal{L}^1(X)$ for nonseparable X with $\mathrm{dens}(X) = \aleph_1$. Assume then the result has been already proved for spaces X with $\mathrm{dens}X < \aleph_\beta$, and let us prove it for spaces X with $\mathrm{dens}X = \aleph_\beta$. Let us set $X = \lim_\to X_\alpha$ as an inductive limit, ordered by ordinals α with $|\alpha| < \aleph_\beta$, of spaces X_α with $\mathrm{dens}X_\alpha = |\alpha|$ as in the diagram.



Let $\phi, \mathcal{L}^1$ be as before. For the first step, consider the diagram



By the induction hypothesis the extension Φ_1 exists. For the general step, the diagram:



shows that one should take as $\mathcal{L}^1(X)$ the inductive limit $\lim_{\rightarrow} \mathcal{L}^1(X_\alpha)$, with embedding J given locally by $J_\alpha u_\alpha$; and how the extension of ϕ through J can be achieved. ∎

Discussion: Uniqueness of the universal Lindenstrauss space. The space $\mathcal{L}^1(X)$ is certainly not unique (we will still see a third construction below). It is however unique in the wider category of superspaces of X (having as objects superspaces $a : X \to A$ of X and as morphisms between two superspaces $a : X \to A$ and $b : X \to B$ an operator $t : A \to B$ such that $ta = b$). As a Banach space "it" has however a dependence upon the finite dimensional decomposition $\{X_n\}$ chosen as starting point. Nevertheless, the universal property of the embedding is enough to ensure the functorial character of the construction, in the sense that given an operator $t : X \to Y$ between two Banach spaces and universal embeddings $x : X \to \mathcal{L}^1(X)$ and $y : Y \to \mathcal{L}^1(Y)$ there exists a unique operator $T : \mathcal{L}^1(X) \to \mathcal{L}^1(Y)$ making commutative the square; i.e., $Tx = yt$. This could be used to transfer properties of X to $\mathcal{L}^1(X)$. It would be nice to know if this construction could yield a space $\mathcal{L}^1(X)$ with FDD when X has it; or, in the nonseparable setting, an $\mathcal{L}^1(X)$ with PRI when X has a PRI.

The simplex S formed by all the Banach limits on l_∞ is often called the Poulsen simplex; and the space of continuous functions on S when endowed with the weak*-topology is called the Gurarij space $\mathcal{G}$ (see [27]). The Gurarij space is universal for separable Lindenstrauss spaces, in the sense that every separable Lindenstrauss space is a complemented subspace of $\mathcal{G}$. It is therefore also true that every separable Banach space admits an embedding into the Gurarij space which is universal with respect to the extension of operators into Lindenstrauss spaces. This yields the following addition to the omnibus lemma: *If Y is separable and $g_Y : Y \to \mathfrak{G}$ denotes an embedding of Y into the Gurarij space then the exact sequence* $0 \to Y \xrightarrow{j} X \to Z \to 0$ *is* $(\lambda, \mathfrak{L})$*-trivial if and only if there exists a weak* continuous map* $\omega : S \to \lambda B_{X^*}$*, such that* ${g_Y}^*{}_{|S} = j^*\omega$. One could be tempted to believe that $\mathcal{L}^1(X)$ could be $C(B_{X^*})$. It is not: the sequence $0 \to \mathcal{G} \to C(B_{\mathcal{G}^*}) \to Q \to 0$ cannot $\mathfrak{L}$-split since $\mathcal{G}$ is not complemented in any $C(K)$-space (see also Proposition 5.1). Nevertheless, $C(B_{X^*})$ and $\mathcal{L}^1(X)$ are not so far away: Semadeni shows in [26] that if a compact space K is the projective limit (in the category of compact spaces and continuous functions) of a filtering family K_i of compact spaces then $C(K)$ is the inductive limit (in the category of Banach spaces) of the family $C(K_i)$. This means that given a Banach space X admitting a representation $X = \overline{\bigcup X_n}$ one has $C(B_{X^*}) = \lim_{\to} C(B_{X_n^*})$; while $\mathcal{L}^1(X) = \lim_{\to} C(B_{P_n^*})$.

A $(1+\varepsilon)$-isomorphic version of the universal construction for the separable case. A useful "isomorphic" or "local" version of the universal Lindenstrauss is also possible; and indeed we already used it in the case of subspaces of c_0 to have a control on the resulting space: observe that starting with a subspace H of c_0 the isometric construction cannot yield c_0 as the final Lindenstrauss superspace (it can be directly seen from the fact the the first step in the isometric construction already contains $C[0,1]$; or using [15, Example 6] of Johnson and Zippin which shows that there is no equal norm extension of $C(K)$-valued operators from subspaces of c_0. The local construction in Theorem 3.1 however got c_0 as the bigger superspace. To get this local construction we should work as follows: start with a representation of the space X as $\overline{\bigcup X_n}$ in which each X_n is finite-dimensional, with embeddings $i_n : X_n \to X_{n+1}$. Next, let us consider a $(1+\varepsilon_1)$-isometric embedding $j_1 : X_1 \to l_\infty^{a(1)}$ and form the push-out space P_1 of i_1, j_1 to obtain a diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{j_1} & l_\infty^{a(1)} \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle u_1} \\ X_2 & \xrightarrow[J_1]{} & P_1. \end{array}$$

Take then a $(1+\varepsilon_2)$-isometric embedding $j_2 : P_1 \to l_\infty^{a(2)}$ and let ω_2 be a homogeneous $(1+\varepsilon_2)$-weak*-continuous selector for j_2. We set $\Phi_2 = {\phi_{p_1}}^{\omega_2}$. The process continues working now with $\delta_2\Phi_2$ and ϕ_3, forming their push out space P_2 as in the diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{j_2} & l_\infty^{a(2)} \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle u_2} \\ X_3 & \xrightarrow[J_2]{} & P_2 \end{array}$$

and so on. The resulting Lindenstrauss superspace is the inductive limit $\lim_{\to} l_\infty^{a(k)}$ with respect to the operators $j_{k+1}u_k : {l_\infty}^{a(k)} \longrightarrow {l_\infty}^{a(k+1)}$. The embedding of X into this inductive limit is locally defined by the operators $j_{k+1}J_k$.

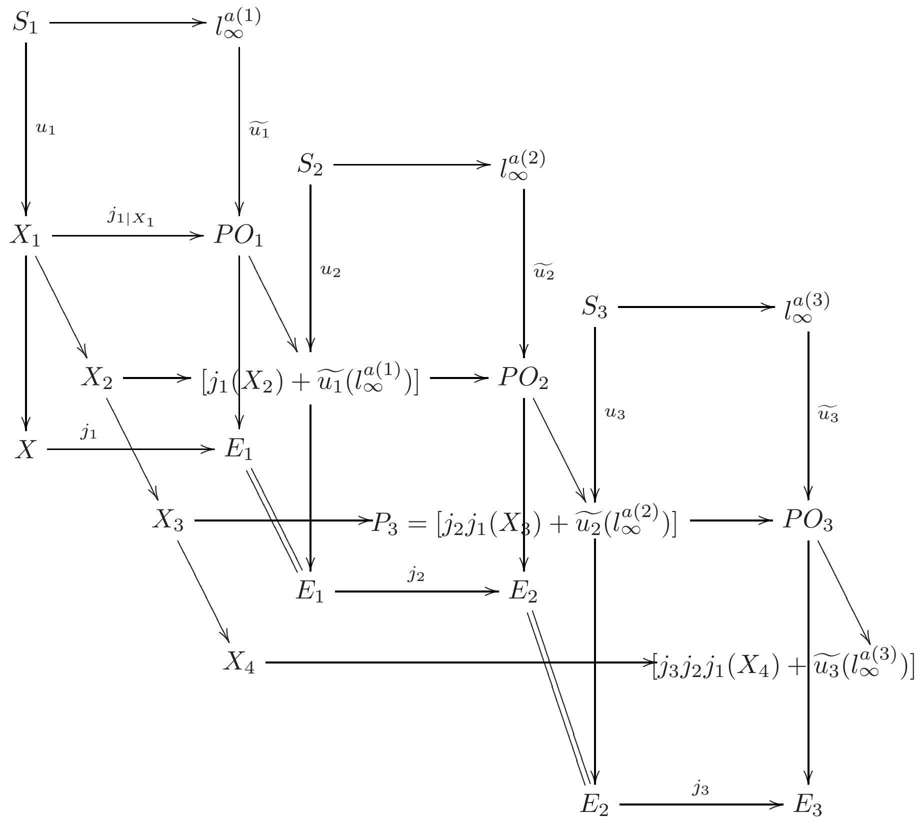
The Bourgain-Pisier sequence. In [5] Bourgain and Pisier showed that every separable Banach space X can be embedded into some $\mathcal{L}_\infty$-space $\mathcal{L}_\infty(X)$ in such a way that the quotient space $\mathcal{L}_\infty(X)/X$ has the Schur and Radon-Nikodym properties. Let us show that this embedding allows one to extend operators into Lindenstrauss spaces.

Proposition 4.2: *For every separable Banach space X and $\varepsilon > 0$ the Bourgain–Pisier sequence*

$$0 \to X \to \mathcal{L}_\infty(X) \to \mathcal{L}_\infty(X)/X \to 0$$

is $(1+\varepsilon, \mathcal{L})$-trivial.

Proof. What we will actually show is that the process followed by Bourgain and Pisier in [5] is an "isomorphic version" of the previous local method. The diagram below might be helpful to understand the first steps in the construction.

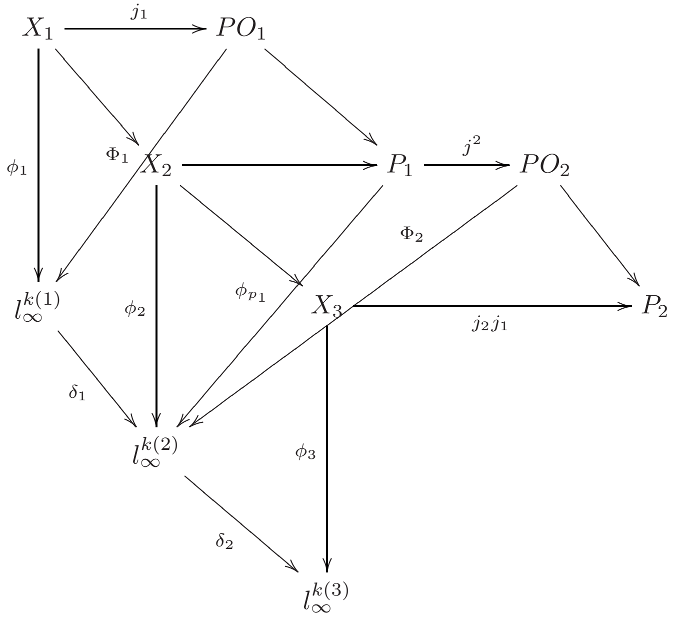


Assume as before that $X = \overline{\bigcup X_n}$ with each X_n finite dimensional, and let $i_n : X_n \to X$ be the inclusion. Bourgain and Pisier use in [5] a clever device to control the resulting $\mathcal{L}_\infty$-space: fix $\lambda > 1$; and then set $\lambda^{-1} < \eta < 1$. Let $s_1 : S_1 \to l_\infty^{a(1)}$ be a subspace such that there is an isomorphism $u_1 : S_1 \to X_1$ with $\|u_1\| \leq \eta$ and $\|u_1^{-1}\| \leq \lambda$. Form the push-out of s_1 and i_1 to obtain a Banach space E_1, an isometric embedding $j_1 : X \to E_1$ and an embedding $\widetilde{u_1} : l_\infty^{a(1)} \to E_1$ making a commutative square, namely, $j_1 i_1 u_1 = \widetilde{u_1} s_1$. We call PO_1 the subspace of E_1 that is the push-out of s_1 and u_1. In this case, PO_1 is λ-isomorphic to $l_\infty^{a(1)}$. Next we form the push-out of the restriction of j_1 to X_1 and the inclusion $X_1 \to X_2$. This new push-out space is $P_2 = [j_1(X_2)+\widetilde{u_1}(l_\infty^{a(1)})]$ (endowed with the norm of E_1).

For the next step, take $s_2 : S_2 \to l_\infty^{a(2)}$ a subspace such that there is an isomorphism $u_2 : S_2 \to P_2$ with $\|u_2\| \leq \eta$ and $\|u_2^{-1}\| \leq \lambda$. Form the push-out of s_2 and the composition $S_2 u_2 : P_2 \to E_1$ that we call momentarily U_2. This yields a Banach space E_2 with an isometric embedding $j_2 : E_1 \to E_2$ and an embedding $\widetilde{u_2} : l_\infty^{a(2)} \to E_2$ making a commutative square, namely: $j_2 U_2 = \widetilde{u_2} s_2$. We call PO_2 the push-out of s_2 and u_2, a subspace of E_2 λ-isomorphic to $l_\infty^{a(2)}$. Form then the push-out of the restriction $j_2 : X_2 \to PO_2$ and the embedding $X_2 \to X_3$. This new push-out space is $P_3 = [j_2 j_1(X_3) + \widetilde{u_2}(l_\infty^{a(2)})]$ (endowed with the norm of E_2), and the process can continue. The resulting $\mathcal{L}_{\infty,\lambda}$ superspace is the inductive limit

$$PO_1 \xrightarrow{j_2\widetilde{u_1}} PO_2 \xrightarrow{j_3\widetilde{u_2}} PO_3 \longrightarrow \cdots$$

while the embedding $j : X \to \mathcal{L}_\infty(X)$ is given by$j(x) = j_n \cdots j_1(x)$ when $x \in X_n$. We show now that this embedding provides a $\mathfrak{L}$-trivial sequence $0 \to X \xrightarrow{j} \mathcal{L}_\infty(X) \to Q \to 0$. The extension process of an operator ϕ from X into a Lindenstrauss space $\mathfrak{L}$ is depicted in the following diagram



Here Φ_1 denotes the unique push-out operator corresponding to the couple $\phi_1 u_1$ and its norm-preserving extension to $l_\infty^{a(1)}$; while ϕ_{p_1} denotes the unique push-out operator corresponding to the couple ϕ_2, Φ_1. For $n > 1$, Φ_n is the unique push-out operator corresponding to the couple $\Phi_{n-1}u_n$ and its norm-preserving extension to $l_\infty^{a(n)}$; while ϕ_{p_n} denotes the unique push-out operator corresponding to the couple ϕ_{n+1}, Φ_n. The desired extension operator Φ of ϕ is thus locally given by

$$\Phi(x) = \Phi_n(x) \quad \text{if} \quad x \in PO_n. \qquad \blacksquare$$

It is plausible that the Bourgain–Pisier embedding is universal with respect to the extension of operators into $\mathcal{L}_\infty$-spaces, however the local approach seems to be useless in this regard.

5. Different types of $\mathcal{L}_\infty$-splitting

In [11], the class of Lindenstrauss–Pełczyński spaces, in short $\mathcal{LP}$-spaces, was introduced as those Banach spaces E such that every E-valued operator defined on a subspace of c_0 can be extended to the whole c_0. In this language, the result formulated by Lindenstrauss and Pełczyński is that Lindenstrauss spaces are $\mathcal{LP}$-spaces. It is shown in [11] that $\mathcal{LP}$-spaces are $\mathcal{L}_\infty$-spaces. Our purpose now is to show that $\mathcal{C}(\mathcal{K})$-splitting , $\mathcal{L}$-splitting, $\mathcal{LP}$-splitting and $\mathcal{L}_\infty$-splitting are different notions.

Proposition 5.1: *$\mathcal{C}(\mathcal{K})$-splitting does not imply $\mathcal{L}$-splitting.*

Proof. It was already observed in [3] that as a consequence of the isometric predual of l_1 that is not complemented in any $C(K)$-space there constructed, the Gurarij space $\mathcal{G}$ (see [13]) cannot be complemented in any $C(K)$-space. Thus, the exact sequence $0 \to \mathcal{G} \to C(B_{\mathcal{G}^*}) \to Q \to 0$ which obviously $\mathcal{C}(\mathcal{K})$-splits cannot $\mathcal{L}$-split. $\blacksquare$

Let us show now how difficult seems to be for an embedding into a Lindenstrauss space to be either $\mathcal{LP}$-trivial or $\mathcal{L}_\infty$-trivial.

Proposition 5.2: *Let X be an infinite dimensional Banach space, that is not itself a Lindenstrauss space. If some embedding $X \to \mathcal{L}^1$ of X into a Lindenstrauss space is $\mathcal{LP}$-trivial, then every complemented subspace of X contains c_0.*

Proof. To prove the first part, let X be a separable Banach space not containing copies of c_0, and let $0 \to X \to \mathcal{L}^1 \to Q^1 \to 0$ be an exact sequence that we assume it $\mathcal{LP}$-splits. Applying the Bourgain–Pisier construction to X we obtain an exact sequence $0 \to X \to \mathcal{L}_\infty(X) \to S \to 0$. The space $\mathcal{L}_\infty(X)$ is an $\mathcal{LP}$-space because (see [11]) it is an $\mathcal{L}_\infty$-space that does not contain c_0. The inclusion $X \to \mathcal{L}_\infty(X)$ cannot be extended to $\mathcal{L}^1$ since Lindenstrauss spaces share with $C(K)$-spaces the property that every operator that is not an isomorphism on a copy of c_0 must be weakly compact (see [14]); in which case the Dunford–Pettis property of $\mathcal{L}_\infty$-spaces yields that such extension must also be completely continuous, hence X should be a reflexive Schur space. ∎

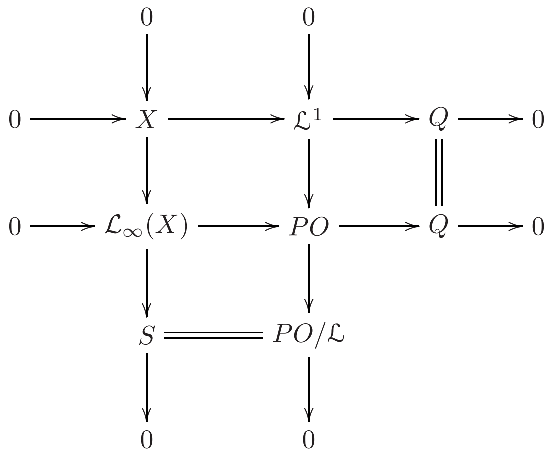
It has been already proved that

Proposition 5.3: *$\mathcal{L}$-splitting does not imply $\mathcal{LP}$-splitting.*

That $\mathcal{LP}$-splitting does not imply $\mathcal{L}_\infty$-splitting was already proved in [11] answering a question of Zippin [31].

Proposition 5.4: *Let $\mathcal{L}^1$ be a Lindenstrauss space, and let $X \to \mathcal{L}^1$ be an embedding. If it is $\mathcal{L}_\infty$-trivial then $\mathcal{L}^1/X$ must be isomorphic to a complemented subspace of a Lindenstrauss space.*

Proof. Consider the completed push-out diagram



in which the first vertical sequence is the Bourgain–Pisier sequence associated to X, which has been shown in Proposition 4.2 that $\mathcal{L}$-splits. Therefore $PO = \mathcal{L}^1 \oplus S$. If $0 \to X \to \mathcal{L}^1 \to Q \to 0$ is $\mathcal{L}_\infty$-trivial then $PO = \mathcal{L}_\infty(X) \oplus Q$. So, Q is a complemented subspace of $\mathcal{L} \oplus S$. Since S is Schur, it is not hard to see that these two spaces are essentially incomparable, following the notation of [12]. Therefore by [12] every complemented subspace of their product can be decomposed as $Q = A \oplus B$ where A is complemented in $\mathcal{L}^1$ and B complemented in S. Now, B must be finite dimensional because a Lindenstrauss space cannot have an infinite dimensional quotient with the Schur property; so Q is isomorphic to a complemented subspace of a Lindenstrauss space. ∎

Unfortunately the previous conditions are far from necessary: see in [7] non-trivial sequences having the form $0 \to C(A) \to C(B) \to C(K) \to 0$.

Conjecture: No sequence $0 \to X \to C(K) \to C(K)/X \to 0$ is $\mathcal{L}_\infty$-trivial.

6. Appendix. Classical proof for Theorem 1

Lindenstrauss and Pełczyński suggest in [20] using the following generalization of Lazar and Lindenstrauss (see [19]) of Edward's separation theorem to prove Theorem 1.1:

Proposition 6.1: *Let Y be a Banach space with $Y^* = L_1(\mu)$ for some μ. Let be $g : B_{Y^*} \to (-\infty, \infty]$ be a concave ω^*-lower semi-continuous function satisfying:*

$$g(y^*) + g(-y^*) \geq 0, \quad y^* \in B_{Y^*}.$$

Let F be a face essentially ω^*-closed of B_{Y^*} and suppose that f is a function over H=conv $(F \cup -F)$ which is ω^*-continuous, affine, symmetric, and such that $f \leq g_{|H}$. Then there exists a ω^*-continuous, affine, and symmetric extension h of f to the whole B_{Y^*} in such a way that $h \leq g$.

Proof of Theorem 1: Without loss of generality we can assume that $\|T\| = 1$. It is enough to prove that for each $\varepsilon > 0$, and for each $y \in c_0 - X$, T can be extended to an operator on $\{X + [y]\}$ having norm at most $1 + \varepsilon$. This means to show that there exists $\xi \in Y$ such that for all $x \in X$ one has

$$\|\xi - Tx\| \leq (1 + \varepsilon)\|y - x\|;$$

in other words, ξ has to satisfy that for all $y^* \in B_{Y^*}$

$$\begin{aligned}\sup_{x\in X} \left(Tx(y^*) - (1 + \varepsilon)\|y - x\|\right) = G(y^*) \leq \xi(y^*) \leq F(y^*)\\ = \inf_{x\in X} \left(Tx(y^*) + (1 + \varepsilon)\|y - x\|\right).\end{aligned}$$

Notice that $F(-y^*) = -G(y^*)$. Consider now $\underline{F}$, a concave function ω^* lower semi continuous dominated by F, then the function $\overline{G}(y^*) = -\underline{F}(-y^*)$ is a convex, ω^*-upper semi-continuous function which dominates $-F$.

Claim: *If $\underline{F}(y^*) + \underline{F}(-y^*) \geq 0$ then the value ξ exists.*

Proof of the claim. : Let $y_0^* \in extB_{Y^*}$ be an extreme point. It is obvious that there exists $\alpha \in \mathbb{R}$ such that $-\underline{F}(-y_0^*) \leq \alpha \leq \underline{F}(y_0^*)$. Let us consider the function $f(\lambda y_0^*) = \lambda\alpha$ on $H = conv(y_0^* \cup -y_0^*) = [y_0^*, -y_0^*]$. Since $\underline{F}$ is concave we have:

$$\theta\alpha + (1-\theta)(-\alpha) \leq \theta\underline{F}(y_0^*) + (1-\theta)\underline{F}(-y_0^*) \leq \underline{F}(\theta y_0^* + (1-\theta)(-y_0^*)).$$

What is the same, $f \leq \underline{F}_{|H}$. Using the Proposition there exists a function $h : B_{Y^*} \to \mathbb{R}$ which is also affine, symmetric and ω^*-continuous, which extends f and still satisfies $h \leq F$. This is the function h we are looking for: recall that a function h over B_{Y^*} is ω^*-continuous, affine and symmetric if and only if $h(y^*) = y^*(y)$ for some $y \in Y$. Since one has $h(y^*) \leq \underline{F}(y^*)$ and $h(-y^*) \leq \underline{F}(-y^*) \quad \forall y^* \in B_{Y^*}$ then $-\underline{F}(-y^*) \leq h(y^*) \leq \underline{F}(y^*)\ \forall y^* \in B_{Y^*}$. In particular,

$$-F(-y^*) \leq -\underline{F}(-y^*) \leq h(y^*) \leq \underline{F}(y^*) \leq F(y^*) \quad \text{for all} \quad y^* \in B_{Y^*}.$$

The point ξ we are looking is the point making $h(y^*) = y^*(\xi)$ for all y^*. ∎

Once the claim has been proved, the rest of the proof of Theorem 1 closely follows the original one of Lindenstrauss and Pełczyński [20]. We include it for the sake of completeness.

Assume that exists $y^* \in B_{Y^*}$ such that $\underline{F}(y^*) < -\underline{F}(-y^*)$. Then, there exist sequences $\{u_n^*\}_{n=1}^\infty, \{v_n^*\}_{n=1}^\infty \in B_{Y^*}$ which are weak*-convergent to y^* and

$$\lim F(u_n^*) < \lim -F(-v_n^*).$$

From the definition of F and G it can be easily deduced the existence of sequences $\{x_n\}, \{z_n\} \in X$, such that

$$\lim[Tx_n(u_n^*) + (1+\varepsilon)\|y - x_n\|] < \lim[Tz_n(v_n^*) - (1+\varepsilon)\|y - z_n\|] \tag{1}$$

Let μ_n, ψ_n be norm preserving extensions of $T^*u_n^*, T^*v_n^* \in X^*$ to ℓ_1 .

Passing to subsequences if necessary, we can assume that $\mu_n^* \xrightarrow{\omega^*} \mu$, and that $\psi_n^* \xrightarrow{\omega^*} \psi$. Then $\mu_{|X} = \psi_{|X} = T^*y^*$, because $T^*u_n^*$ weak*-converges to T^*y^*

as $\mu_{n|X}$ weak*-converges to $\mu_{|X}$ and $T^*v_n^*$ weak*-converges to T^*y^* as $\psi_{n|X}$ weak*-converges to $\psi_{|X}$.

A well-known property of the ω^* convergence in ℓ_1 is that, for every $u \in \ell_1$, and every weak*-null sequence w_n one has

$$\lim_{n\in\mathbb{N}} (\|w_n + u\| - \|w_n\| - \|u\|) = 0.$$

So,

$$\lim_{n\in\mathbb{N}} (\|\mu_n\| - \|\mu_n - \mu\| - \|\mu\|) = 0 \tag{2}$$

$$\lim_{n\in\mathbb{N}} (\|\psi_n\| - \|\psi_n - \psi\| - \|\psi\|) = 0$$

If we restrict to X, since $\|\mu_n\| = \|T^*y_n^*\|, \|\psi_n\| = \|T^*y_n^*\|$, while the norm of the other functionals in (2) just decrease, we get, say, $\|\mu\| = \|\psi\| = \|T^*y^*\| = r$, and therefore

$$\limsup \ \|\mu_n - \mu\| \leq 1 - r \quad \limsup \ \|\psi_n - \psi\| \leq 1 - r.$$

Notice that $\limsup\, (\mu_n - \mu)(y) = 0$, $\limsup\, (\psi_n - \psi)(y) = 0$, and thus

$$\begin{aligned}
\lim \quad & [Tx_n(y_n^*) + (1+\varepsilon)\|y - x_n\| - Tz_n(y_n^*) + (1+\varepsilon)\|y - z_n\|] \\
&= \lim \quad [\mu_n(x_n) - \psi_n(z_n) + (1+\varepsilon)(\|y - z_n\| + \|y - x_n\|)] \\
&= \lim \quad [(\mu_n - \mu)(x_n) - (\psi_n - \psi)(z_n^*) \\
&\quad + T^*y^*(x_n - z_n) + (1+\varepsilon)(\|y - z_n\| + \|y - x_n\|)] \\
&= \lim \quad [(\mu_n - \mu)(x_n - y) - (\psi_n - \psi)(z_n - y) \\
&\quad + T^*y^*(x_n - z_n) + (1+\varepsilon)(\|y - z_n\| + \|y - x_n\|)] \\
&= \lim \quad [S_1(n) + S_2(n) + S_3(n)]
\end{aligned}$$

Here

$$\begin{aligned}
S_1(n) &= (\mu_n - \mu)(x_n - y) + (1 - r + \varepsilon)\|y - x_n\| \\
S_2(n) &= -(\psi_n - \psi)(z_n - y) + (1 - r + \varepsilon)\|y - z_n\| \\
S_3(n) &= \ T^*y^*(x_n - z_n) + r(\|y - z_n\| + \|y - x_n\|).
\end{aligned}$$

Now, $S_1(n)$ and $S_2(n)$ are nonnegatives for large n, while $S_3(n) \geq 0$ for all n. This contradicts (1) and concludes the proof of Theorem 1. ∎

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